A Symmetry Break in Dynamo Models

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In accordance with the concepts of dynamo theory, large-scale magnetic fields of many astrophysical objects are associated with the rotation-induced break of mirror symmetry [1]. The rotation of objects and the presence of depth density gradients $(\nabla \rho)$ turn out to be sufficient for the existence of a correlation between the vertical velocity and the vorticity of small-scale vortices. This correlation results in the nonzero kinetic helicity $\chi = \langle \mathbf{v} \cdot \text{rot} \mathbf{v} \rangle$, where $\mathbf{v}$ is the turbulent velocity and $\langle \cdot \rangle$ implies averaging over an ensemble. In the simplest cases, it is considered that, in the same hemisphere, the helicity is of identical sign: in the northern hemisphere, it is negative when the density increases with depth, which is natural for self-gravitating bodies. In the southern hemisphere, the helicity is positive, e.g., $\chi = -\cos \vartheta$ [2, 3], where $\vartheta$ is the angle with respect to the rotation $z$ axis in the spherical coordinate system $(r, \vartheta, \varphi)$. The existence of the nonzero helicity $\chi$ of the hemispheres significantly simplifies the generation of a large-scale magnetic field. At the same time, the helicity is obtained as a result of averaging and is subjected to random fluctuations [4], which, in the long run, affects the generation of the magnetic field [5]. The source of the fluctuations can be either an insufficient number of vortices over which averaging is performed or the existence of anomalies at the object boundaries, e.g., the appearance of tidal forces in binary stars or the presence of thermal inhomogeneities at boundaries of the liquid core and the mantle in planets [6]. These arguments can lead to the break of antisymmetry $\chi$ with respect to the equatorial plane. In the framework of the Parker model for the $\alpha\omega$ dynamo, we show below that, even for a linear mode, a small nonparity ($\chi$) violation can lead to a significant symmetry break of the magnetic field in opposite hemispheres.

We now consider the Parker model often applied to describe processes of the magnetic-field generation in various astrophysical objects [1, 7, 8]. We below make use of the following equations in a thin spherical shell:

$$\frac{\partial \mathbf{A}}{\partial t} = \alpha \mathbf{B} + \mathbf{A}'', \quad \frac{\partial \mathbf{B}}{\partial t} = D \Omega \mathbf{A}' + \mathbf{B}''. $$

Here, $\mathbf{A}$ and $\mathbf{B}$ are azimuthal components of the magnetic-field vector potential ($\mathbf{B} = \text{rot} \mathbf{A}$) of the magnetic field $\mathbf{B}$; $\alpha$ and $\Omega$ are, respectively, the $\alpha$ effect and the effect of the differential rotation both depending on the angular coordinate $\vartheta$, $D$ is the dynamo number proportional to the product of the amplitudes for the $\alpha$ and $\Omega$ effects. Primes indicate derivatives with respect to $\vartheta$. At the poles, $\vartheta = 0$; for $\pi$, the boundary conditions $\mathbf{A} = \mathbf{B} = 0$ are valid. In the simplest case, $\alpha = -(\tau/3) \chi$ [9], where $\tau$ is the characteristic time of vortex rotation. The derivation of Eq. (1) involves averaging over both the azimuthal coordinate $\varphi$ and the radius $r$.

The solution to Eq. (1) is of the form

$$(\mathbf{A}, \mathbf{B}) = e^{\gamma t}(A(\vartheta), B(\vartheta)), $$

which corresponds to the problem of proper values:

$$\gamma \mathbf{A} = \mathbf{A}' $$

$$\gamma \mathbf{B} = D \Omega \mathbf{A}' + \mathbf{B}' $$

Here, $\gamma$ is the complex growth rate, and the amplitudes $A$ and $B$ are also complex. We now introduce the latitude $\theta = 90^\circ - \vartheta$. In accordance with general concepts, the pseudoscalar quantity $\alpha(\theta)$ is antisymmetric relative to $\theta$ with respect to the equator: $\alpha(-\theta) = -\alpha(\theta)$. Therefore, the solutions can be divided in two classes, namely, the dipole one ($D$), $B_{\perp}(-\theta) = -B_{\perp}(\theta)$, $B(\theta) = -B(\theta)$ and the quadrupole solution ($Q$), $B_{\perp}(-\theta) = B_{\perp}(\theta)$, $B(\theta) = B(\theta)$, where $B_{\perp}$ is the radial component of the magnetic field.

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We consider below the dependence of the first three proper values \( \gamma_1, 2, 3 \) and proper solutions \((B_r, B)_{1, 2, 3}\) on the dynamo number for the most propagated distributions \( \alpha \) and \( \Omega \): \( \alpha = \cos \vartheta + \zeta \) and \( \Omega = \sin \vartheta \), the parity of arising solutions, and the dependence of the result obtained on the constant \( \zeta \) that characterizes the helicity asymmetry.

The behavior of the complex proper value \( \gamma(\mathcal{D}) \) for \( \zeta = 0 \) is illustrated by Figs. 1a and 1b. We analyze the real-valued part \( \Re \gamma \). Its positive values correspond to growing solutions. In the figures, the first three solutions with the maximum growth rates are presented. The solutions are sorted for each \( \mathcal{D} \) so that denotation 1 always corresponds to the most rapidly growing mode and, then, denotations 2 and 3 follow.

For positive \( \mathcal{D} \), the first arising mode is steady (the imaginary part of the growth rate is \( \Im \gamma = 0 \), Fig. 1b) and quadrupole (see below). An increase in \( \Re \gamma \) results...
in the fact that the growth rate \( \mathcal{D} \) of the second oscillating dipole mode advances the quadrupole one. The former mode becomes leading, and both modes coexist within a certain interval of values of \( \mathcal{D} \). In Fig. 1b, the symmetric branches of the imaginary part of \( \Im\gamma(\mathcal{D}) = -\Im\gamma(-\mathcal{D}) \) for oscillation frequencies correspond to complex-conjugate solutions. For the oscillating mode, the wave motion from the equator towards the poles is characteristic. The further increase in \( \mathcal{D} \) results in the fact that the steady quadrupole solution ceases to grow, but a new quadrupole oscillating mode (the solutions ordered with a decrease in \( \Im\gamma \)) appears. In this case, the waves also propagate towards the poles.

Fig. 2. (a, b) Oscillating dipole solution and (c, d) steady quadrupole solution for \( \mathcal{D} = 2 \times 10^2 \) and \( \mathcal{C} = 0 \); (e–h) the same for \( \mathcal{C} = 0.1 \) (in Fig. 1, 1 corresponds to a dipole and 2 to a quadrupole). For \( \mathcal{C} = 0.1 \), the spread of the butterfly diagram and the oscillation suppression are characteristic for the southern hemisphere. The steady quadrupole solution weakly depends on \( \mathcal{C} \).
Fig. 3. (a, b) Oscillating quadrupole solution; (c, d) oscillating dipole solution; and (e, f) steady dipole solution for $\mathcal{C} = -10^3$ and $\mathcal{C} = 0$; (g–l) the same for $\mathcal{C} = 0.1$. There is a correspondence to denotations $1$, $2$, and $3$ in Fig. 1.
For $\mathcal{D} < 0$, the first arising solution is a steady dipole. Further, an oscillating quadrupole appears for which the wave solution propagates along the direction from the poles towards the equator, as is the case for the Sun. The growth rate for the quadrupole becomes higher than that for the dipole (Fig. 1a). For large negative values of $\mathcal{D}$, three modes coexist: the oscillating dipole one, the quadrupole mode with close values of $\gamma$, and the considerably weaker steady dipole mode. For long times, the last mode can introduce asymmetry of the hemispheres.

Except for certain negligible details, the appearance of asymmetry of $\mathcal{C} \neq 0$ does not introduce significant changes in the behavior of $\gamma$. The most remarkable difference is the mode splitting for large $\mathcal{D}$: the imaginary parts cease to be identical for different modes, and the denotations $1$ and $2$ in Fig. 1d already do not coincide for different modes. Below, we show that the form of the proper solutions turns out to be much more sensitive with respect to the perturbation $\alpha$.

We consider the butterfly diagram $(\hat{B}_r, \hat{B})(\Theta, t) = \Re(e^{\mathcal{D}t}(\mathcal{C} \equiv B_r, B))$.\footnote{Further, for the sake of simplicity, we omit the sign $\wedge$.} The in-pair comparison of the oscillating modes for which $\mathcal{C} = 0$ and $\mathcal{C} = 0.1$ results in the following: the $\mathcal{C}$ asymmetry introduced to the $\alpha$ distribution significantly changes the form of the proper solution. The structure clearness becomes lost. An even more important fact is that the solution asymmetry for opposite hemispheres appears. As is clearly seen in Figs. 2e and 2f, the oscillating dipole mode is considerably weaker in the southern hemisphere than in the northern hemisphere. It is interesting that for the oscillating quadrupole not decreasing near the equator, we can observe the penetration of the dynamo wave from one hemisphere to the other (see Fig. 3g). For better demonstration of asymmetry, we present the dependence of the maximum values of $B_{\max}(\Theta)$ when $\mathcal{C} = 0$ and $\mathcal{C} = 0.1$ (Fig. 4). In the case of oscillations with a zero average value, $B_{\max}$ is a positive-sign function symmetric with respect to the equator for $\mathcal{C} = 0$ (Figs. 4a, 4c). Introducing a perturbation of $\alpha$ for $\mathcal{D} > 0$ results in a decrease in the dipole oscillating mode for the southern hemisphere and, at the same time, the quadrupole solution remains invariable and steady (Fig. 4b).

For $\mathcal{D} < 0$, the situation is the opposite: introducing a perturbation results in weakening the quadrupole oscillating mode in the southern hemisphere and weakening the dipole oscillating mode in the northern hemisphere (Fig. 4d).
It is easy to understand that this asymmetry of the hemispheres can be obtained by imposing dipole and quadrupole modes. The bases for this suggestion do exist. We consider the perturbed solution \((A + a, B + b)\) of the set of Eqs. (3), which is caused by the perturbation \(\alpha: \alpha = \alpha_0 + \delta \alpha\). Upon substituting the solution into Eqs. (3), we arrive at \(\gamma a = \alpha_0 b + \delta \alpha B + a\). Insofar as in our case, \(\delta \alpha = '\ell\) corresponds to the quadrupole form of symmetry, and \(A\) and \(B\) have different forms of symmetry, we obtain that the perturbation \((a, b)\) is of the opposite forms of symmetry with respect to the unperturbed solution \((A, B)\).

In the other words, the perturbation of the dipole solution \((B_\alpha, B)\) generates the quadrupole one \((b_\alpha, b)\) and vice versa. As a result of the weakening of the magnetic field in one hemisphere and the amplification of the field in the other hemisphere, we deal with the interference of the original field and its growing perturbation (Figs. 2a, 2b, 2e, 2g, 3a, 3b, 3g, 3h, 3c, 3d, 3j, 3k). As is seen in the figures, the effect appears for unsteady solutions. For nonoscillating modes, \(\delta \gamma = 0\), the solution is stable (Figs. 2c, 2d, 2g, 2h, 3e, 3f, 3k, 3l).

Thus, we have considered, as an example, the dynamo model often used in theory to describe generation of the magnetic field. We have shown that even negligible variations of relevant parameters (sometimes significantly smaller than the observation errors) can make it possible to attain rather noticeable asymmetry of the magnetic-field generation in different hemispheres. It is evident that in more complicated nonlinear models, in which the magnetic field affects the \(\alpha\) effect, the \(\delta \alpha\) perturbation can be attained self-consistently in the course of solving the problem. The effect described in the present paper of the dynamo-wave penetration from one of the hemispheres to the other remains at present as previously the object of intense studies (see [10] for details).

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REFERENCES


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