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PHYSICS

Instabilities in Dynamos

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According to the dynamo theory, the magnetic field in convective conducting media is generated through the conversion of kinetic energy into magnetic energy [1, 2]. The magnetic field generation is a threshold phenomenon: provided that the velocity field has the "right" configuration and possesses a sufficient energy (the magnetic Reynolds number is larger than its critical value, $R_m > R_m^{cr}$), an exponentially growing solution arises. The magnetic field grows until it begins to exert the reverse action on the velocity field. This action is not reduced to a simple decrease in the intensity of motions but is accompanied by a change in the shape of the spectra for the velocity and magnetic fields, given the requirements imposed by the energy and helicity conservation laws [3]. At the same time, the flow velocity itself can remain very high, as suggested by the huge R_m in astrophysics [1]. Moreover, it is interesting that the velocity field taken from the nonlinear problem (where the magnetic field has already ceased to grow indefinitely) is capable of generating an exponentially growing magnetic field in the case of a kinematic dynamo if the action of the magnetic field on the flow is neglected [4–7]. In other words, the stability criteria for the complete system of dynamo equations, including the induction equation for the magnetic field, the Navier–Stokes equation, and the Lorentz force, differ from those for the linear kinematic dynamo problem for the magnetic field with the velocity field taken from the nonlinear dynamo problem: when the first solution is stable, the second solution can be unstable. We will consider this effect for the 3D problem of a dynamo in a plane layer heated from the bottom, explain the effect using a simple galactic dynamo model for a thin disk (or, after a slight modification, in a thin spherical shell) as an example, and discuss some of the consequences for planetary dynamo problems.

Consider the dynamo equations for an incompressible fluid ($\nabla \cdot \mathbf{V} = 0$) in an infinite layer $0 \le z \le 1$ heated from the bottom that rotates with an angular velocity Ω about the vertical z axis. Introducing the following units of measurement for velocity $\mathbf{V}$, time t, pressure P, and magnetic field $\mathbf{B}$: κ/L, L^2/κ, $\rho\kappa^2/L^2$, and $\sqrt{2\Omega\rho\kappa\mu_0}$, where L is the unit of length, κ is the thermal diffusivity, ρ is the fluid density, μ_0 is the magnetic constant, we will write the system of dynamo equations in Cartesian coordinates (x, y, z) as [8, 9]

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{V} \times \mathbf{B}) + q^{-1}\Delta\mathbf{B},$$

$$E\mathrm{Pr}^{-1}\left[\frac{\partial \mathbf{V}}{\partial t} - \mathbf{V} \times (\nabla \times \mathbf{V})\right]$$

$$= \mathrm{curl}\mathbf{B} \times \mathbf{B} - \nabla P - \mathbf{1}_z \times \mathbf{V} + \mathrm{Ra}\,T\mathbf{1}_z + E\Delta\mathbf{V}, \qquad (1)$$

$$\frac{\partial T}{\partial t} + (\mathbf{V} \cdot \nabla)(T + T_0) = \Delta T,$$

$$\frac{\partial \hat{\mathbf{B}}}{\partial t} = \mathrm{curl}(\mathbf{V} \times \hat{\mathbf{B}}) + q^{-1}\Delta\hat{\mathbf{B}}.$$

The last equation for the field $\hat{\mathbf{B}}$ is equivalent to the induction equation for the magnetic field $\mathbf{B}$, with the only difference being that $\hat{\mathbf{B}}$ exerts no action on the velocity field $\mathbf{V}$.

The dimensionless Prandtl, Ekman, Rayleigh, and Roberts numbers are specified in the following form: $\mathrm{Pr} = \nu/\kappa$, $E = \nu/2\Omega L^2$, $\mathrm{Ra} = \alpha g_0 \delta T L/2\Omega\kappa$, and $q = \kappa/\eta$, where ν is the kinematic viscosity, α is the volume expansion coefficient, g_0 is the gravity, δT is the perturbation in temperature T relative to the "diffusive" (nonconvective) temperature distribution $T_0 = 1 - z$, and η is the magnetic diffusivity. System (1) is closed by periodic boundary conditions along the horizontal. For the boundaries $z = 0$ and 1, we use zero values for

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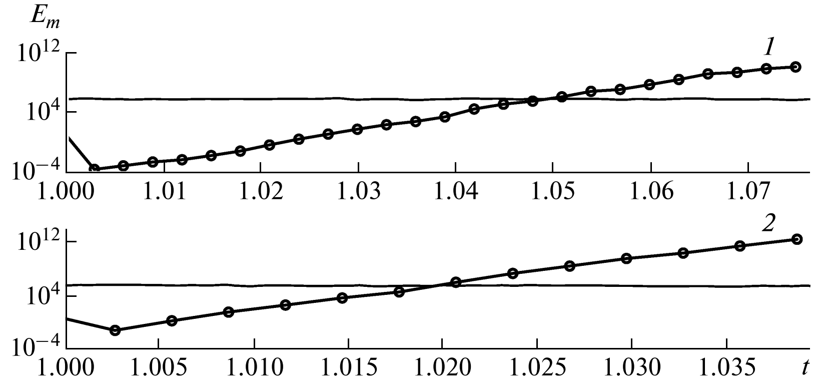

Fig. 1. Evolution of the magnetic energy E_m (line) and energy $\hat{E}_m$ (dots) for the regimes with Ra = 4×10^2, $q = 10$ (*1*) and Ra = 1×10^3, $q = 3$ (*2*). In both regimes, Pr = 1 and $E = 2 \times 10^{-5}$.

the perturbations in temperature $T = 0$. Given the chosen T_0 profile, this is equivalent to specifying the temperatures at the boundaries: $\mathcal{T} = T + T_0 = 1$ and 0. For the velocity field, we take the nonpenetration condition and the gradients of the tangential components to be zero at $z = 0$ and 1: $V_z = \frac{\partial V_x}{\partial z} = \frac{\partial V_y}{\partial z} = 0$. For the magnetic field **B** (and $\hat{\mathbf{B}}$), we use pseudo-vacuum boundary conditions: $B_x = B_y = \frac{\partial B_z}{\partial z} = 0$. The computations were performed on 64^3 grids.

Figure 1 presents the behavior of the magnetic energy $E_m = B^2/2$ for the fields **B** and $\hat{\mathbf{B}}$. Despite the use of the same velocity field **V** in both induction equations for the magnetic field, the behavior of the energy is different: the energy remains at some quasi-stationary level for **B** and grows exponentially with time for $\hat{\mathbf{B}}$. It remains to assume that the space–time synchronization of the velocity and magnetic fields is very important. For $\hat{\mathbf{B}}$, this synchronization is definitely smaller, because there is no Lorentz force for this field. Similar results were obtained in [6] for the problem of a dynamo in a layer with an induced force. Consider what happens in more detail using a simple galactic dynamo model as an example, which allows the analytical apparatus to be used.

One of the simplest galactic dynamo models is the 1D model for a thin disk [10]:

$$\frac{\partial A}{\partial t} = \alpha B + A'', \qquad \frac{\partial B}{\partial t} = -\mathcal{D}A' + B'', \tag{2}$$

where A and B are the azimuthal components of the vector potential of the magnetic field ($\mathbf{B} = \mathrm{curl}\, A$) and the magnetic field **B**, α is the hydrodynamic helicity dependent on the z coordinate over the disk thickness, and $\mathcal{D}$ is the dynamo number proportional to the product of the amplitudes of the $\hat{\alpha}$ and $\hat{\omega}$ effects. The primes correspond to the derivative with respect to z. The following vacuum boundary conditions are met at the disk boundaries $z = \pm 1$: $B = 0$ and $A' = 0$. We will seek a solution in the form

$$(A, B) = e^{\gamma t}(\mathcal{A}(z), \mathcal{B}(z)). \tag{3}$$

System (2) can then be reduced to an eigenvalue problem:

$$\gamma\mathcal{A} = \alpha\mathcal{B} + \mathcal{A}'', \qquad \gamma\mathcal{B} = -\mathcal{D}\mathcal{A}' + \mathcal{B}'', \tag{4}$$

where γ is the growth rate. According to general views, the pseudo-scalar quantity $\alpha(z)$ is antisymmetric in z: $\alpha(-z) = -\alpha(z)$. In this case, the solutions can be broken down into two classes: dipole (D) $\mathcal{A}(-z) = \mathcal{A}(z)$, $\mathcal{B}(-z) = -\mathcal{B}(z)$ and quadrupole (Q) $\mathcal{A}(-z) = -\mathcal{A}(z)$, $\mathcal{B}(-z) = \mathcal{B}(z)$. This is equivalent to the problem for one half of the disk $z \in [0, 1]$ with the following boundary conditions for $z = 0$: $\mathcal{A}' = 0$, $\mathcal{B} = 0$ (D) and $\mathcal{A} = 0$, $\mathcal{B}' = 0$ (Q).

For $\mathcal{D}_- < 0$, the first mode ($\mathcal{D}_-^{cr} \sim -8$) is a quadrupole and does not oscillate, $\Im\gamma = 0$. For $\mathcal{D} > 0$, the first mode is a dipole and oscillates, $\Im\gamma \neq 0$, while the generation threshold is considerably higher, $\mathcal{D}_+^{cr} \sim 200$.

Introducing nonlinearity in the form

$$\alpha = \frac{\alpha_0(z)}{1 + E_m}, \tag{5}$$

where $E_m = (B^2 + A'^2)/2$ is the magnetic energy, allows us to obtain solutions stationary for Q and quasi-stationary for the D type of symmetry, in general, without changing significantly the form of the eigensolutions of system (4).

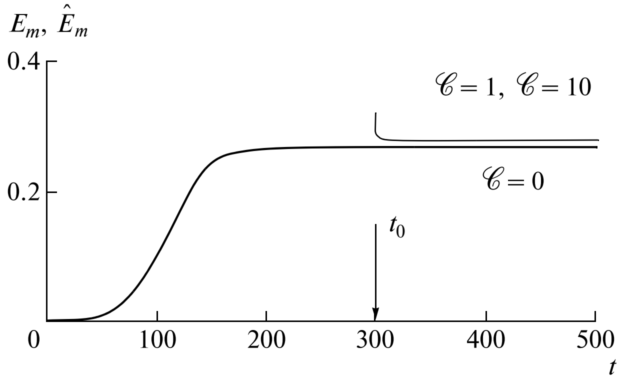

Fig. 2. Evolution of the magnetic energy E_m for $t < t_0$ of system (2), (5) for $\mathscr{D} = -10$. At $t_0 = 300$, the generation of a "new" magnetic field $(\hat{A}, \hat{B})$ with the initial conditions defined by the constant $\mathscr{C}$ begins. The plots of E_m and $\hat{E}_m$ (with $\mathscr{C} = 0$) for $t > t_0$ coincide. All solutions are stationary.

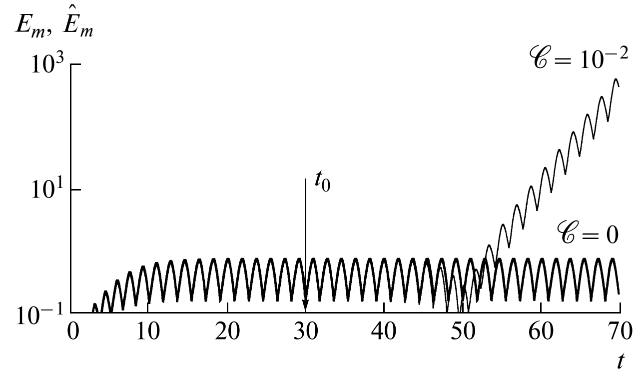

Fig. 3. Evolution of the magnetic energy E_m for $t < t_0$ and $\mathscr{D} = 300$. At $t_0 = 30$, the generation of a "new" magnetic field $(\hat{A}, \hat{B})$ begins. The plots of E_m and $\hat{E}_m$ (with $\mathscr{C} = 0$) for $t > t_0$ coincide. For $\mathscr{C} \neq 0$, after the transition process, the phase shift between E_m and $\hat{E}_m$ increases, which gives rise to an exponential growth of $\hat{E}_m$.

By analogy with (1), we will add the equations for $(\hat{A}, \hat{B})$ with the same helicity $\alpha = \alpha(z, E_m)$, where α does not depend on the fields $(\hat{A}, \hat{B})$, to system (2). Next, simultaneously integrating the equations for (A, B) and $(\hat{A}, \hat{B})$ numerically for negative values[1] of $-\mathscr{D} \in (10\text{–}1000)$, we find that both pairs of fields are stationary (see Fig. 2). However, depending on the initial conditions, $\hat{E}_m$ can take on various values. In the model, we took an initial condition in the form $(\hat{A}, \hat{B}) = (A, B)(1 + C\varepsilon)$, where ε is a random variable in the interval $(-0.5, 0.5)$ and C is a constant. The difference in stationary levels, of course, follows from the linearity and homogeneity of the system of equations for $(\hat{A}, \hat{B})$. Therefore, the solution was determined within an arbitrary factor.

For positive $\mathscr{D}$, the generation threshold is higher, $\mathscr{D} \sim 200$, while the nonlinear solution is oscillatory. In this case, just as for system (1), the field $(\hat{A}, \hat{B})$ begins to grow indefinitely (see Fig. 3). The emerging instability resembles the instabilities that emerged in [5, 6]. Another fact, which is important from our viewpoint, is the appearance of a delay of the field $(\hat{A}, \hat{B})$ relative to (A, B): $\theta \approx -\pi/3$. Interestingly, averaging E_m in (5) over time (so that α becomes stationary) leads to the disappearance of the instability. The question arises as to whether the instability is a consequence of the field periodicity in time or is caused by other factors.

[1] For our Galaxy, it is commonly assumed that $\mathscr{D} = -10$.

The stability of the solutions of system (4) is known to be closely related to the behavior of the solutions for $\mathscr{D} = -\mathscr{D}$ [11]. Let us represent the perturbed solution of system (4) as $(\tilde{A}, \tilde{B}) = (\mathscr{A} + a, \mathscr{B} + b)$, where $(\mathscr{A}, \mathscr{B})$ satisfies the steady-state solution of system (4), (5) and (a, b) is the perturbation with the same boundary conditions. Substituting $(\tilde{A}, \tilde{B})$ into (4) and expanding α into a series of B,[2] $\alpha = \alpha_0 + \frac{\partial\alpha}{\partial B} b$, we will then obtain a system for (a, b):

$$\begin{aligned} \gamma a &= \alpha^e b + a'', \\ \gamma b &= -\mathscr{D} a' + b'', \end{aligned} \qquad (6)$$

where $\alpha^e = \alpha + \frac{\partial\alpha}{\partial B} B$. For nonlinearity (5), we have

$$\alpha(z)^e = \frac{1 - B^2}{(1 + B^2)^2}\alpha_0 \sim -\frac{\alpha_0}{B^2} \quad \text{for} \quad |B| \gg 1. \qquad (7)$$

The behavior of the $\alpha\omega$-model (4) is determined by the sign of $\mathscr{D}\alpha$, and its change when passing to the perturbed problem (6) turns out to be fundamentally important. As a result, the problem for the perturbation (a, b) is reduced to the problem with an effective dynamo number $\mathscr{D}^e = -\mathscr{D}/B^2$.

The stability of the field $(\hat{A}, \hat{B})$ for $\mathscr{D} < 0$ can then be explained as follows. The solution $(\hat{A}, \hat{B})$ is finite and stationary, because the generation threshold $\mathscr{D}_+^{cr}$ for (4) for positive $\mathscr{D}$ is much higher than $\mathscr{D}^e$, $\mathscr{D}_+^{cr} \gg \mathscr{D}^e$.

[2] In the $\alpha\omega$-dynamo models, it is tacitly assumed that $B \gg A'$.

The determinacy of the field $(\hat{A}, \hat{B})$ within a factor corresponds to equalization of the vector fields (A, B) and $(\hat{A}, \hat{B})$ relative to each other.

Interestingly, the solution of systems (2), (5) is stationary in the interval $-\mathcal{D} \in (10, ..., 10^3)$, although $\mathcal{D}_+^{cr}$ for the quadrupole oscillatory mode for $\mathcal{D} > 0$ is ~200. This is because the solution tends to the regime of a "strong" magnetic field $B \sim \mathcal{D}^{1/2}$ in such a way that $\alpha \sim 1/\mathcal{B}^2$; as a result, $\mathcal{D}^e$ remains close to its value at the generation threshold.

For positive $\mathcal{D}$, the field (A, B) and, hence, $\alpha(B)$ oscillate and additional information about the correlation of the fields is required. Instead of (7), we have $\alpha^e \sim -\frac{\alpha_0 \hat{B}}{|B|^3}$. If the phase shift θ between B and $\hat{B}$ is small, then the behaviors of the fields (A, B) and $(\hat{A}, \hat{B})$ with time coincide. However, our calculations (see Fig. 3) show that θ becomes significant with time: $\theta \approx \pi/3$. This resembles the situation that arises during the appearance of a parametric resonance, when α is modulated by a signal with a frequency $\Omega \sim 2\omega$, $\omega = \Im\gamma$. This assumption is confirmed by the fact that the instability disappears when α is averaged over time. Note that, on the whole, the pattern of the appearance of an instability remains as before when periodic boundary conditions are used instead of quadrupole ones.

Consider how the delay θ of the field $(\hat{A}, \hat{B})$ relative to (A, B) affects the generation of $\hat{A}^2 + \hat{B}^2$ near the generation threshold $\mathcal{D}_+^{cr}$. Let us write the linear system in complex form:

$$i\omega\hat{A} = \alpha\hat{B} - k^2\hat{A},$$
$$i\omega\hat{B} = -i\mathcal{D}_+^{cr}k\hat{A} - k^2\hat{B}. \tag{8}$$

It follows from the condition for solvability of (8), $(k^2 + i\omega)^2 = -i\mathcal{D}_+^{cr}k\alpha_0$ with $\alpha = \alpha_0$, that $\omega^2 = k^4 = 1$. It also follows from a linear analysis that the phase shift between $\hat{A}$ and $\hat{B}$ is $\varphi = \pm\pi/4$. For the nonlinear problem, $\varphi \longrightarrow \pm\pi/2$, which corresponds to the maximum value of $\hat{A}$ when $\hat{B}$ is zero.

Taking the solutions in the form of waves, $B = b\sin(x - t)$, $\hat{A} = \sin(x - t + \varphi + \theta)$, $\hat{B} = \sin(x - t + \theta)$, and $\alpha = 1/(1 + B^2)$, we will obtain the dependence of the field generation relative to θ. Since the equation for $\hat{B}$ does not include the initial field (A, B), we will consider only the generation of $\hat{A}^2$. Multiplying the equation for $\hat{A}$ by $\hat{A}$, we will then obtain $\delta\hat{A}(\varphi, \theta) = \alpha_0 \int_0^{2\pi} \frac{\hat{B}\hat{A}}{1 + B^2} dt$. If $|\Pi| > 1$, where $\Pi = \frac{\delta\hat{A}(\varphi, \theta)}{\delta\hat{A}(\varphi, 0)}$, then $(\hat{A}, \hat{B})$ is unstable. The expression for Π is

$$\delta\Pi(\varphi, \theta) = 1 + \frac{h_2}{h_1}\tan\varphi,$$
$$h_1 = \frac{\cos\theta^2(4 - 32^{1/2}) - 2(2^{1/2} - 1)}{2^{1/2}(2^{1/2} - 1)} \approx 1 - 0.3\cos\theta^2,$$
$$h_2 = -\frac{\sin(2\theta)(32^{1/2} - 4)}{2^{3/2}(2^{1/2} - 1)} \approx -0.8\sin(2\theta). \tag{9}$$

If $\theta = 0$, then $b_2 = 0$ and $\delta\hat{A}(\varphi, 0) = h_1 = 2^{1/2} - 4$. Π is then singular for $\varphi \to \pm\frac{\pi}{2}$, which corresponds to the appearance of an instability.

Summarizing the results for the stationary and nonstationary regimes, we have the following stability criteria for the fields $(\hat{A}, \hat{B})$. For $\mathcal{D} < 0$, (A, B) is stationary and $(\hat{A}, \hat{B})$ is unstable if $\left|\frac{\mathcal{D}}{\mathcal{D}_+^{cr}B^2}\right| \gg 1$. When (A, B) oscillates, $(\hat{A}, \hat{B})$ continues to oscillate with (A, B) with increasing phase shift. Subsequently, a parametric resonance leading to an indefinite growth of $(\hat{A}, \hat{B})$ can appear.

Since the nonlinearities considered are very general in character, the above ideas also have far-reaching consequences for several other dynamo models.

The linear analysis of the axisymmetric equations for an $\alpha\omega$-dynamo in a sphere gives the following [12]: at positive dynamo numbers (typical of the Earth) in the presence of a meridional velocity U_p, the first mode is a dipole one (D) with $\Im\gamma = 0$. A decrease in the intensity of meridional circulation leads to the transition to an oscillatory quadrupole mode (Q) and to an increase in the critical dynamo number. For negative $\mathcal{D}$ at $U_p \neq 0$, the first mode is a quadrupole one with $\Im\gamma = 0$ and a decrease in U_p leads to the transition to an oscillatory dipole mode and also to a decrease in $\mathcal{D}^{cr}$.

The decrease in U_p for $\mathcal{D} > 0$ corresponds to the change of the regime of rare reversals to the regime of frequent geomagnetic field reversals [13]. The appearance of meridional circulation at $\mathcal{D} < 0$ leads to the destruction of the dipole field often associated with the Maunder minimum on the Sun. Since there is no large asymmetry in the generation thresholds, the models in question must preferably give an unstable solution for the field $\hat{\mathbf{B}}$.

A similar situation also arises for multimode dynamo models at large magnetic Reynolds numbers: $\hat{\mathbf{B}}$ will grow, which also corresponds to the results of calculations for a 3D dynamo in a plane layer with an induced force [4–7], cascade models [6], also with an induced force, and the calculations presented in Fig. 1 in the problem with heating from the bottom.

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