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Effect of Compressibility on the Generation of Hydrodynamic Helicity in the Earth's Liquid Core

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Abstract—By the example of a standard three-dimensional model of thermal convection in a rotating spherical shell, it is shown how radial density gradient $\nabla\rho$ affects the generation of hydrodynamic helicity χ. For example, the generation of χ weakly depends on $\nabla\rho$ inside a Taylor cylinder and, outside the cylinder, the compressibility of the liquid amplifies the generation of χ. The consequences for the geodynamo theory are considered.

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1. INTRODUCTION

Since the mid-1990s, there has been a significant breakthrough in the theory of the planetary geodynamo. Having started from successful models of the geodynamo (Schubert, 2007), scientists moved on to modeling magnetic fields that are observable in the solar system from other planets and their satellites (Stanley and Glatzmaier, 2009). It is well known that the existence of planetary magnetic fields at a distance that is comparable in its scale with the region of their generation is connected with the rapid rotation of the planets. For example, in the absence of rotation, the generated magnetic fields have a rope-shaped structure (Kazantsev, 1967; Meneguzzi and Pouquet, 1989) and, due to fast decreasing with distance, become almost invisible far from the generation region. At the same time, the spatial spectrum of planetary fields decreases with an increase in the wavenumber. The role of rotation is reduced to both the symmetrization of the toroidal magnetic field around the axis of rotation and the pumping of the field into the axial-symmetric mode and to the destruction of the properties of the mirror symmetry of the hydrodynamic turbulence (Parker, 1955; Moffatt, 1978). The destruction of the symmetry of a small-scale velocity field ($\mathbf{v}$) leads to the appearance of the hydrodynamic helicity $\chi = \langle \mathbf{v} \cdot \mathrm{rot}\mathbf{v} \rangle$, where $\langle \ldots \rangle$ denotes averaging. The χ quantity is closely connected with the well-known α-effect, which describes the generation of large-scale magnetic fields by small-scale turbulence (Krause and Rädler, 1980). It is evident that, at a slow rotation, there are already no reasons to expect a coincidence between the direction of the geomagnetic dipole (averaged over several millions years) with the axis of rotation; at the same time, as it is shown by numerical experiments, one should not even expect the predomination of the dipole mode at all.

Turning back to the question about the generation of helicity, it should be noted that the appearance of a correlation between the velocity and its vorticity can be ensured by different mechanisms. In particular, this can be shear flows (for details, see (Hollerbach and Rüdiger, 2004)), when helicity is generated near solid boundaries where the velocity gradients in the boundary layers are large.

Another scenario of χ generation is more commonly encountered in astrophysics. It is based on the idea of the passage of a convective vortex in a rotating compressible medium. In the presence of a negative density gradient with respect to the height, an ascending (descending) vortex expands (contracts) and begins to rotate more slowly (rapidly), ensuring negative helicity in one hemisphere and positive helicity in the other (Parker, 1955). It is evident that this mechanism can operate far from boundaries and probably turns to be more effective at small scales where the memory about the boundaries of the volume is already lost and the shear flows are small. The situation is complicated by the possibility of the transfer of χ over the spectrum (Frik, 2010). Note that this mechanism is completely impossible in Boussinesq models, for which the velocity field satisfies the condition of zero divergence $\nabla \cdot \mathbf{V} = 0$. At the same time, it is well known that, for the liquid core of the Earth, the increase in the density with respect to the radius is only 20% (for more details regarding the parameters that are important for the geodynamo, see (Braginsky and Roberts, 1995)), and a priori it is quite not evident whether the compressibility has a significant effect on helicity generation. We recall that the interest in anelastic models (Braginsky and Roberts, 1995; Glatzmaier and Roberts, 1996) taking into account the compressibility was initiated by the correct formulation of a thermodynamic problem, not by questions regarding the principle of magnetic field generation.

Below, by the example of a three-dimensional model of thermal convection in a rotating spherical shell, we consider how the existence of the radial density gradient changes the generation of helicity and what can be expected when the obtained results are extrapolated to the regime of the Earth's dynamo.

2. CONVECTION EQUATIONS

Let us consider a spherical layer ($r_1 \le r \le r_0$) of compressible liquid with density $\rho(r)$ rotating around the z axis with angular velocity Ω, where (r, θ, φ) is the spherical coordinate system (for the Earth, $r_1 = 0.35$ and $r_0 = 1$). We neglect the $\frac{\partial\rho}{\partial t}$ terms and introduce the following units of measurement for velocity $\mathbf{V}$, time t, and pressure P: κ/L, L^2/κ, and $\rho\kappa^2/L^2$, where L is the unit of length, κ is the coefficient of molecular thermal conductivity, and $\bar{\rho}$ is the profile averaged density; after that, we write the equations of thermal convection in the form

$$\begin{gathered}\mathrm{Pr}^{-1}\mathrm{E}\rho\frac{\partial\mathbf{V}}{\partial t} - \mathbf{V}\times\nabla\times\rho\mathbf{V} - \mathbf{V}^2\nabla\rho \\ = -\nabla \mathrm{P} + \mathrm{F} + \mathrm{E}\rho\Big(\nabla^2\mathbf{V} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{V})\Big), \\ \rho\frac{\partial T}{\partial t} + \rho(\mathbf{V}\cdot\nabla)(T + T_0) = \nabla^2 T, \\ \mathrm{F} = -\rho\mathbf{1}_z\times\mathbf{V} + \mathrm{Ra}\rho Tr\mathbf{1}_r.\end{gathered} \tag{1}$$

The dimensionless Prandtl and Ekman numbers and the modified Rayleigh number are introduced as $\mathrm{Pr} = \frac{\nu}{\kappa}$, $\mathrm{E} = \frac{\nu}{2\Omega L^2}$, and $\mathrm{Ra} = \frac{\alpha g_0\delta TL}{2\Omega\kappa}$, where ν is the coefficient of kinematic viscosity; α is the coefficient of volume expansion; g_0 is the free-fall acceleration; and δT is the unit of the perturbation of temperature T relative to the T_0 equilibrium profile.

Problem (1) is closed by the boundary conditions for $r = r_i$, r_0. Temperature T was perturbed using the zero-boundary conditions, which, together with the profile $T_0 = \frac{r_i/r - r_0}{r_0 - r_i}$, given above, corresponds to a fixed value of total temperature $T_0 + T$: (1, 0) at the boundaries. For the $\mathbf{V}$ velocity field, zero-boundary conditions at the external boundary $r = r_0$ are used. At the boundary with the internal core, $r = r_i$, $V_r = 0$, and $V_\theta = 0$. The azimuth velocity component $V_\varphi = \omega s$ is calculated from the angular momentum equation for the internal solid core rotating around the vertical z axis with the angular velocity ω under the action of viscous forces:

$$\begin{gathered}I\mathrm{Pr}^{-1}\frac{\partial\omega}{\partial t} = r_i^4\oint\frac{\partial}{\partial r}\Big(\frac{V_\varphi}{r}\Big)_{r = r_i}\sin^2\theta d\theta d\varphi, \\ I = \frac{8}{15}\pi r_i^5,\end{gathered} \tag{2}$$

where I is the moment of inertia of the solid core relative to the z axis and (s, φ, z) is the cylindrical coordinate system.

Problem (1) was solved using the expansion in spherical functions and Chebyshev polynomials (in the radial direction) (Simitev, 2004). The multiprocessor code was implemented in FORTRAN-95 using the MPI libraries (Reshetnyak, 2011). The calculations were carried out on 128^3 meshes.

3. GENERATION OF HELICITY

The properties of convection at rapid rotation have been studied in many works (see the references in (Reshetnyak, 2010)). With an increase in the Rayleigh numbers as Pr ~ 1, the first appearing convective modes are cyclonic vertical columns outside the Taylor cylinder (TC) (the region above/under the solid core). In the zero approximation (without regard to the viscosity and Archimedean forces), the columns satisfy the Taylor–Proudman theorem $\frac{\partial\mathbf{V}}{\partial z} = 0$ in the main volume. Such flows are called geostrophic and the pressure gradient is balanced by the Coriolis force in them. Let us consider the northern hemisphere. The vortex direction of the liquid in a column is determined by whether the overheated flow is ascending ($V_z > 0$, $T > 0$) or cold and descending ($V_z < 0$, $T < 0$). If a column is formed near the solid boundary, its transverse cross section decreases and this results in the appearance of flow vorticity in the column $\omega = \mathrm{rot}\mathbf{V}$ (Fig. 1). The correlation between vertical velocity V_z and rotation in the horizontal plane ($\omega_z \ne 0$) leads to the appearance of thehydrodynamic helicity χ (Fig. 2).

An increase in the intensity of thermal sources initiates convection inside the TC where the higher threshold of convection excitation is connected with the fact that the vertical flows are deflected from the z axis of rotation at an almost right angle near the boundaries. In this case, the radius of flow vorticity by the Coriolis force is minimal and, as a consequence, the losses owing to dissipation are maximal. The distribution of the χ helicity inside the TC has a more complicated structure; namely, it is alternating—positive near r_i and negative near r_0. For the southern hemisphere, $\chi(-z) = -\chi(z)$.

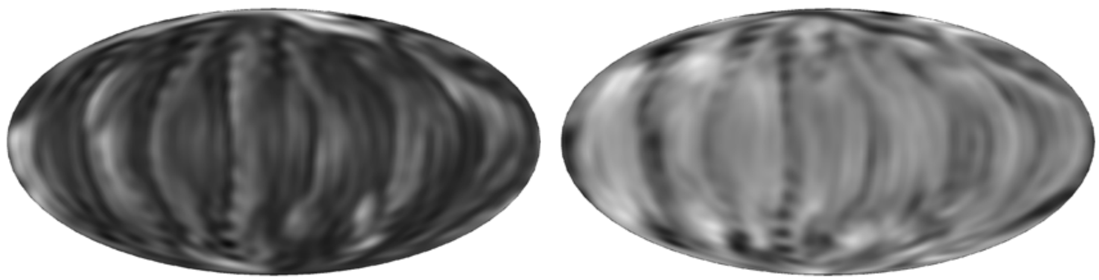

Fig. 1. Radial cross-section of (a) the perturbation of temperature T (−0.09, 043) and (b) azimuth velocity V_φ (−372, 360) for $r = 0.95$, $\mathrm{E} = 2 \cdot 10^{-4}$, Pr = 1, and $\mathrm{Ra} = 8 \cdot 10^2$. The values in brackets correspond to the range of the fields. The dark color corresponds to negative values.

Let us consider the sources of helicity χ in more detail. It follows from $\frac{\partial \mathbf{V}}{\partial t} \sim \mathrm{PrE}^{-1}(\mathbf{1_z} \times \mathbf{V} + \mathrm{Ra}T\mathrm{r})$ that

$$\frac{\partial \chi}{\partial t} \sim \mathrm{PrE}^{-1}\Big(\frac{\partial}{\partial z}\frac{\langle V^2\rangle}{2} - \langle V_2 \nabla \cdot \mathbf{V}\rangle + [(\nabla \times \mathbf{V}) \times \mathbf{V}]_z + \mathrm{Ra}[T\mathbf{r} \cdot \mathrm{rot}\mathbf{V} + \mathrm{rot}(T\mathbf{r}) \cdot \mathbf{V}]\Big). \qquad (3)$$

In an incompressible medium ($\nabla \cdot \mathbf{V} = 0$), the first term in the right-hand side of (3), which is caused by the Coriolis force and the term in the Rayleigh number, which is connected with buoyancy, are responsible for the generation of helicity. Figure 2 presents the helicity distributions for different Ra. For small Ra, the flows are close to geostrophic ones ($\frac{\partial E_K}{\partial_z} = 0$ even at the equator, and $E_K = V^2/2$ is the kinetic energy) and the kinetic energy changes at the boundary $r = r_0$. This change generates negative helicity in the northern hemisphere and positive helicity in the southern hemisphere (see Fig. 2a). It is interesting that a similar result is also obtained for not viscous boundary conditions (Reshetnyak, 2010) where the tangent component of velocity is already nonzero. The fundamental difference between two the types of boundary conditions is the zero value of χ at the boundary for nonviscous boundary conditions and a nonzero value for viscous ones. The estimation of the generation source in an Ekman layer with a thickness of $\sim\mathrm{E}^{1/2}$ yields $\mathrm{E}^{-1/2}/E_K$. As for the order of magnitude, the convective deflection of large-scale helicity χ from the boundary to the main volume is $\mathcal{G} \sim E^{-1/2}E_K\mathrm{E}V = \mathrm{E}^{1/2}V^3$. Note that this mechanism does not ensure helicity generation in the main volume. Another interesting feature is the fact that the horizontal scale of the column $l_\perp \sim E^{-1/3}$ does not appear in the expression for $\mathcal{G}$. An increase in Ra does not change the helicity distribution qualitatively outside the TC.

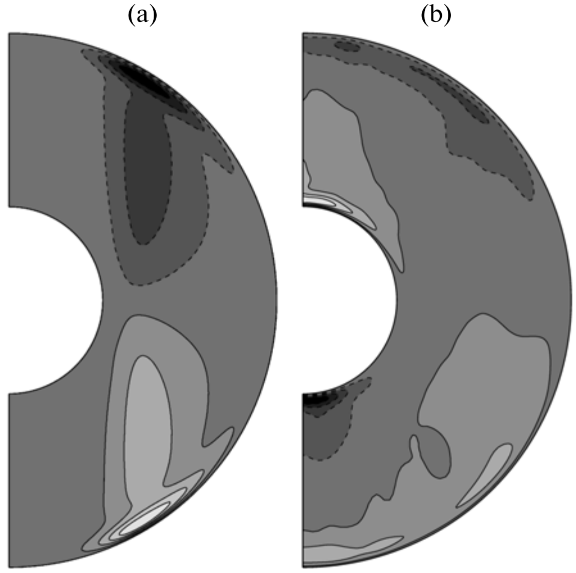

Fig. 2. Meridian cross-section of hydrodynamic helicity χ for incompressible liquid for $\mathrm{E} = 2 \cdot 10^{-4}$ and Pr = 1; (a) $\mathrm{Ra} = 1.5 \cdot 10^2$ (−38000, 38000) and (b) $\mathrm{Ra} = 8 \cdot 10^2$ ($-1.53 \cdot 10^6$, $1.62 \cdot 10^6$). The dashed isolines correspond to negative values.

Let us consider the behavior of the terms connected with the Archimedean force. These terms become effective with an increase in Ra, when convection also appears inside the TC. In contrast to the convection outside the TC, the horizontal scale of the flows inside the TC is large and the axial-symmetric velocities are on the order of magnitude of higher harmonics. For the northern hemisphere, the first term at the Ra number is positive near r_j and negative near r_0. To estimate the second term at Ra, we use the relationship $\mathrm{rot}(T\mathbf{r}) \cdot \mathbf{V} = (\nabla T \times \mathbf{r}) \cdot \mathbf{V}$. Let us introduce a local cylindrical coordinate system (s, φ, z) with the z axis along the axis of the column. Then, $\langle(\nabla T \times \mathbf{r}) \cdot \mathbf{V}\rangle = -\langle\nabla_s T_z V_\varphi\rangle > 0$; i.e., both Archimedean terms operate synchronously and ensure the change in the helicity sign in each of the hemispheres. It is important to note that, in addition to the difference in the spatial dependences of the helicity inside and outside the TC, the physical meaning of χ is also different in these regions in the sense of the dynamics of average fields. If outside the cylinder we have an average value by the columns and the scale separation (Krause and Rädler, 1980) that is required for the theory is valid at least in the perpendicular direction, for the region inside the TC, such a separation does not in fact exist and the main contribution to the generation of helicity is made by large-scale flows. Such a difference can be a basis for creating simple models of the geodynamo, e.g., Parker models, with different spatial localizations of the α- and ω-effects.

If the values of the correlation between velocity $\mathbf{V}$ and vorticity ω are large, the third term in the right-hand side is small, but it can lead to instabilities of the form $\frac{\partial}{\partial t}$ ($|\mathbf{V}||\omega| \cos\phi) \sim |\mathbf{V}|\,|\omega|\sin\phi$, where ϕ is the angle between vectors $\mathbf{V}$ and ω. For stationary $|\mathbf{V}|$ and $|\omega|$, we have a linear growth in the angle: $\phi \sim -t$. Note that the cross-product $\mathbf{V} \times \boldsymbol{\omega}$ equals the nonlinear convective term in (1) up to a factor and is responsible for the transfer of kinetic energy over the spectrum.

Let us turn back to the term that is connected with compressibility. Using $\nabla \cdot (\rho(r)\,\mathbf{V}) = 0$, we have $-\langle V_z \nabla \cdot \mathbf{V} \rangle = \mathrm{C}\langle V_z V_r\rangle$, where $\mathrm{C} = \frac{1}{\rho}\frac{d\rho}{dr}$; i.e., the sign of χ is negative for the northern hemisphere and positive for the southern hemisphere; therefore, the sign of the source coincides with that of $\mathcal{G}$, but the generation of χ occurs across the whole volume of the liquid core, not only at the boundary. The effect of compressibility leads to more uniform generation of χ across the whole volume and, as a consequence, to more efficient generation of the magnetic field by small-scale turbulence. Figure 3 presents cross-sections of χ for (a) $\rho_0/\rho_i = 0.81$ (close to the Earth's regime) and (b) with a large ρ_0 gradient: $\rho_0/\rho_i = 0.36$. An enlargement of the scale of χ is well observed with an increase in $\nabla\rho$.

4. ESTIMATES FOR THE EARTH: DISCUSSION

Let us present estimates of the characteristic values for the Earth's liquid core (Jones, 2000). Assuming that the Coriolis and Archimedean forces are balanced, $\alpha g_0 \delta\, T \sim 2\Omega V_{\omega d}$, where $V_{\omega d} = 3 \cdot 10^{-4}$ m s^{-1} is the velocity of the westward drift of the magnetic field; for $L = 2.26 \cdot 10^6$ m^2 s^{-1} and $\kappa = 10^{-5}$ m^2 s^{-1}, we have $\mathrm{Ra} = \frac{\alpha g_0 \delta \mathrm{T} L}{2\Omega\kappa} \approx 1.5 \cdot 10^7$. Then, in a dimensionless form, $\mathrm{Ra}T \sim V$. For the developed turbulence, $V \sim V_r \sim V_0 \sim V_\varphi$. The estimate for the Earth's core of compressibility yields $C = -0.2$.

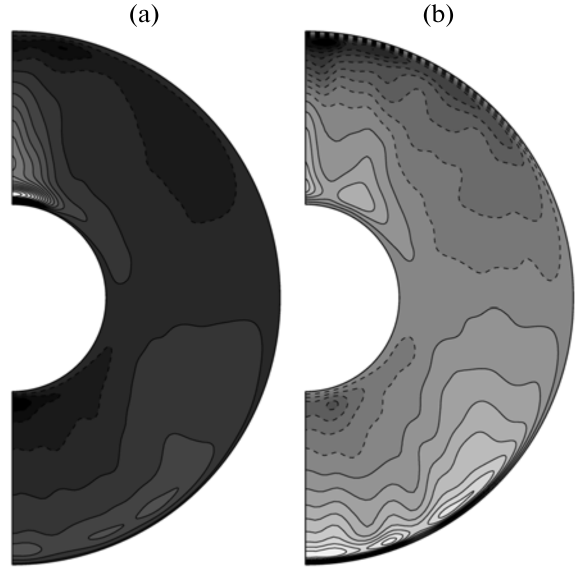

Fig. 3. Meridian cross-section of hydrodynamic helicity χ with allowance for compressibility for $\mathrm{E} = 2 \cdot 10^{-4}$, $\mathrm{Pr} = 1$, and $\mathrm{Ra} = 8 \times 10^2$: (a) $(-10^6, 10^6)$, $\rho = 1.148 - 0.237r^2$ and (b) $(-7.4 \cdot 10^5, 7.4 \cdot 10^5)$, $\rho = 1.714 - 1.144r^2$.

Let us estimate the effectiveness of helicity generation due to compressibility effects. First, we consider the region outside the TC: $\Pi = \frac{2C\langle V_r V_z\rangle}{\frac{d\langle V^2\rangle}{dz}} \sim 2\mathrm{CE}^{-1/2}/V$. Taking $\mathrm{E} = 10^{-15}$ for the Earth and estimating the dimensionless velocity by the velocity of the westward drift $V = V_{\omega d}L/\kappa = 7 \cdot 10^7$, we obtain $\Pi \approx 1$; i.e., the compressibility has an effect on helicity generation. Moreover, since the signs of the helicity sources coincide, the helicity generation is amplified outside the TC. Note that at small scales far from the solid boundaries $\frac{d\langle V^2\rangle}{dz} \to 0$ and without regard to the convective deflection of the helicity from the boundaries $|\Pi| \to \infty$, which does not correspond to the real contribution of the compressibility to the generation of χ.

For the region inside the TC, assuming that $\mathbf{r} \cdot \mathrm{rot}\mathbf{V} \sim \mathbf{z} \cdot \mathrm{rot}\mathbf{V} \sim 2mzV_\varphi/s$, we have $\Pi = \frac{C\langle V_r V_z\rangle}{2zm\mathrm{Ra}\langle TV_\varphi\rangle/s} \sim \frac{C_s}{2zm}$. The estimate for Π yields $|\Pi| \ll 1$, where we took into account that the azimuth wavenumber $m > 1$ and $s < z$. In other words, in the liquid core of the Earth, the compressibility effects must not have a significant effect on the helicity generation inside the TC.

5. CONCLUSIONS

In this work, it is shown that the compressibility effects have a significant effect on the generation of hydrodynamic helicity in the liquid core of the Earth. In turn, the presence of hydrodynamic helicity leads to a more efficient generation of the large-scale planetary magnetic field. In addition, we show the difference between the spatial distributions of the hydrodynamic helicity for compressible and incompressible media in a rotating spherical shell.

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