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PHYSICS

The Alpha-Effect in the Presence of Rapid Rotation

M. Yu. Reshetnyak

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Abstract—The relation of the hydrodynamic helicity and alpha-effect predicted by the mean-field dynamo theory is considered using the three-dimensional dynamo model in the rapidly rotating flat layer heated from below. Despite the theoretical limitations associated with the requirement of uniformity and isotropism of random fields, the alpha-effect is actually proportional to helicity in the geostrophic flows under consideration. Interpretation of this nice agreement is proposed.

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The existence of a time-independent magnetic field in nature is most often associated with dynamo processes. According to the dynamo theory, if some requirements on the flows of the conducting liquid are fulfilled [1], the kinetic energy of the flows can transform into the magnetic field energy. This process is accompanied by the dissipation of the magnetic field. If the generation of the magnetic field is more effective than the dissipation, the magnetic field starts to rise to a certain level determined by the convection intensity, the shape of flows, the medium conductivity, and boundary conditions. The inverse effect of the magnetic field on the conductive medium leads to stopping the rise of the magnetic field, and the system approaches the steady or, more often, quasi-steady level. The characteristic time of approaching the quasi-steady level by the system is not prolonged; it is inversely proportional to the degrees of the magnetic Reynolds number. Precisely these quasi-steady states of the systems are correlated to the magnetic fields observed in astrophysics, geophysics, and technical facilities.

In many applications, Reynolds numbers are large and the system is in a turbulent state with extended spatio-temporal spectra. For example, for the liquid core of the Earth this is $N_d \sim 8$ orders of magnitude, and for the convective shell of the Sun $N_d \sim 11$. Detailed description of the structure of turbulent fields requiring the information volumes of $\sim N_d^3$ is impossible by virtue of the limited modern computational powers. However, this is not always necessary since we are interested in the average characteristics of the fields in many cases. Moreover, in some cases, these characteristics can be observed, for example, in the case of remoteness of the observer from the generation zone of the magnetic field.

Below, we consider that the evaluation of the α-effect, which is responsible for the turbulence generation in the mean-field dynamo theory [2], agrees with the three-dimensional calculations in the rotating periodic planar layer heated from below. Although the model flows are close to those used in modern planetary dynamo models [3], these results are also applicable to other rotating objects, where the average hydrodynamic helicity and α-effect can manifest themselves.

Let us consider the dynamo equations for the incompressible liquid ($\nabla \cdot \mathbf{V} = 0$) in the infinite layer $0 \le z \le 1$ rotating with angular velocity Ω relative to vertical axis **z**. Introducing the following measurement units for velocity **V**, time t, pressure P, and magnetic field **B**, $\frac{\kappa}{L}$, $\frac{L^2}{\kappa}$, $\frac{\rho\kappa^2}{L^2}$, and $\sqrt{2\Omega\rho\kappa\mu}$, where L is the unit length, κ is the coefficient of molecular thermal conductivity, ρ is the substance density, and μ is the magnetic susceptibility, let us write the set of dynamo equations in the Cartesian coordinate system (x, y, z) in the form

$$\frac{\partial \mathbf{A}}{\partial t} = \mathbf{V} \times \mathbf{B} + q^{-1}\Delta\mathbf{A}, \quad \mathbf{B} = \operatorname{rot}\mathbf{A},$$

$$\mathrm{E} \cdot \mathrm{Pr}^{-1}\left[\frac{\partial \mathbf{V}}{\partial t} - \mathbf{V} \times (\nabla \times \mathbf{V})\right] \tag{1}$$

$$= \operatorname{rot}\mathbf{B} \times \mathbf{B} - \nabla P - \mathbf{1}_z \times \mathbf{V} + \mathrm{Ra}\, T\mathbf{1}_z + \mathrm{E}\Delta\mathbf{V},$$

Schmidt Joint Institute of Physics of the Earth,
Russian Academy of Sciences,
Bol'shaya Gruzinskaya ul. 10, Moscow, 123810 Russia
e-mail: m.reshetnyak@gmail.com

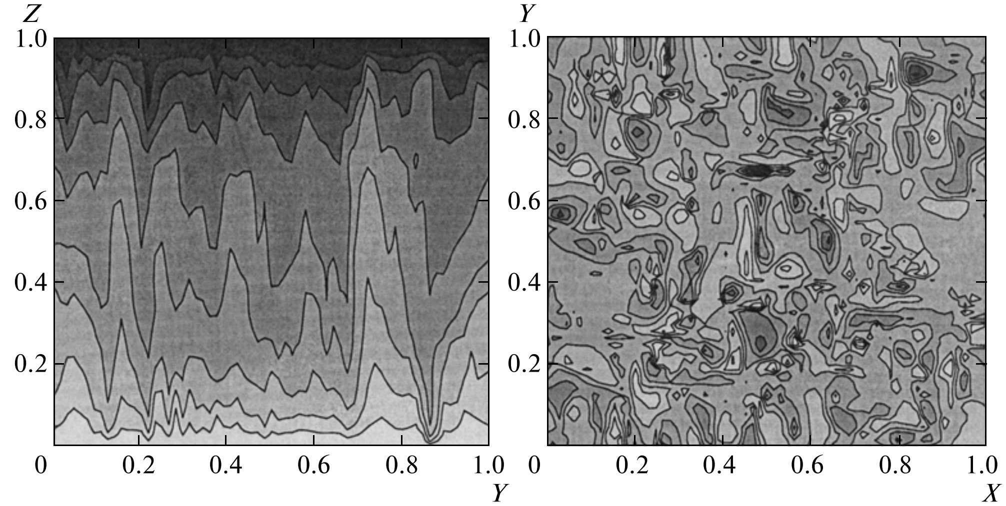

Fig. 1. Sections of the temperature field $\mathcal{T}$ for $x = 0.156$ (to the left), $z = 0.164$ (to the right), and $\mathrm{E} = 10^{-4}$, $\mathrm{Pr} = 1$, $\mathrm{Ra} = 300$, and $q = 2$. The field range (0, 1) and (0.34, 0.91), respectively. The lighter tint corresponds to more heated regions.

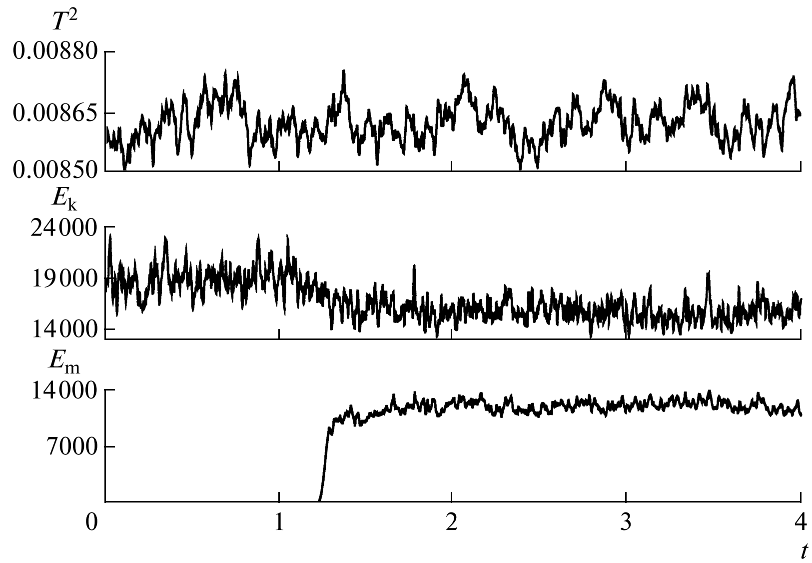

Fig. 2. Evolution of the kinetic energy E_k and magnetic energy E_m averaged by the volume of the quadrate of temperature fluctuations T^2.

$$\frac{\partial T}{\partial t} + (\mathbf{V} \cdot \nabla)(T + T_0) = \Delta T.$$

Dimensionless Prandtl, Ekman, Rayleigh, and Roberts numbers are specified in the form

$$\mathrm{Pr} = \frac{\nu}{\kappa}, \quad \mathrm{E} = \frac{\nu}{2\Omega L^2}, \quad \mathrm{Ra} = \frac{\alpha g_0 \delta T L}{2\omega\kappa}, \; q = \frac{\kappa}{\eta},$$

where ν is the coefficient of kinematic viscosity, α is the thermal expansion coefficient, g_0 is the acceleration of gravity, δT is the perturbation unit of temperature T relative to the diffusion (not convective) temperature distribution $T_0 = 1 - z$, and η is the magnetic diffusion coefficient.

Set (1) is closed by periodic boundary conditions along the horizontal. For boundaries $z = 0, 1$, the zero values are used for temperature perturbations $T = 0$, which is equivalent to specifying the temperatures at the boundaries $\mathcal{T} = T + T_0 = 1, 0$ allowing for the selected profile T_0. For the velocity field, let us accept the nonentry condition and equality to zero of gradients of tangential components at $z = 0, 1$: $V_z = \frac{\partial V_x}{\partial z} =$

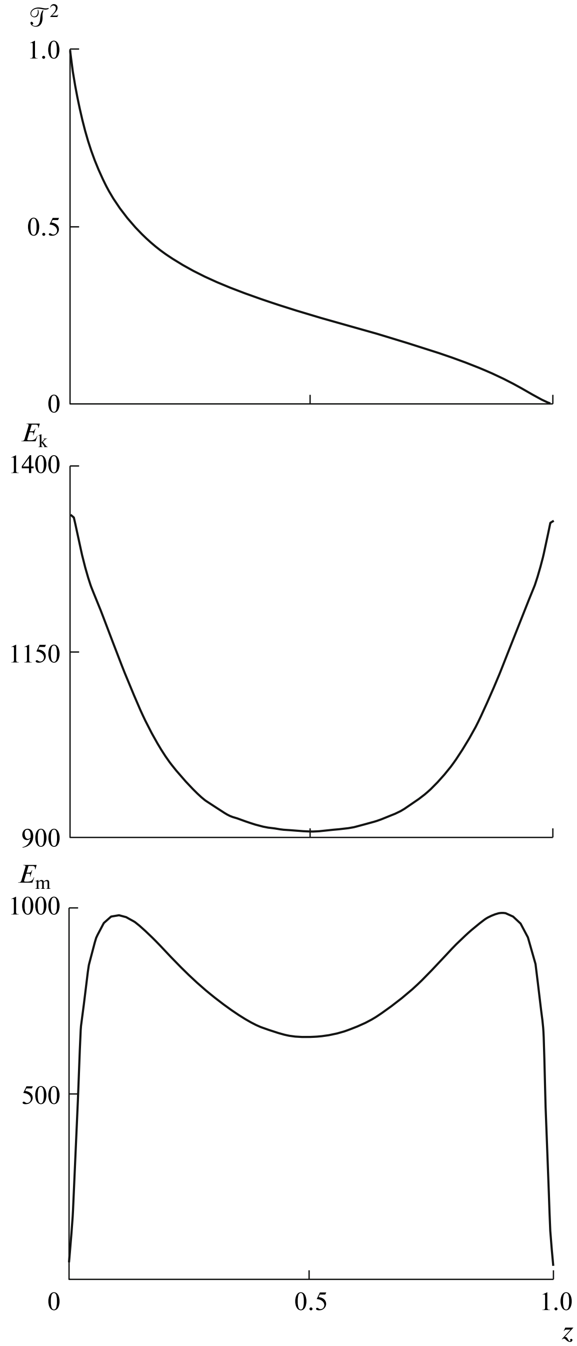

Fig. 3. Distribution profiles of the quadrate of temperature $\mathcal{T}^2$, kinetic energy E_k, and magnetic energy E_m over z.

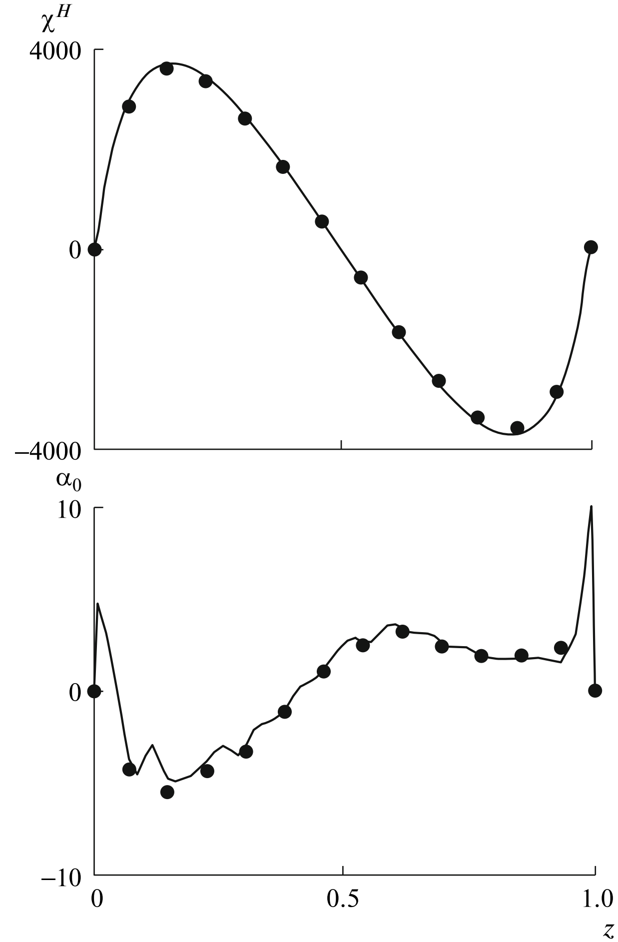

Fig. 4. Profiles of hydrodynamic χ^H and α_0. The solid line corresponds to $\chi^H = \mathbf{V} \cdot \text{rot}\mathbf{V}$ and $\mathcal{E}_i = \mathbf{V} \times \mathbf{B}$, while points correspond to $\chi^H = \mathbf{v} \cdot \text{rot}\mathbf{v}$ and $\mathcal{E}_i = \mathbf{v} \times \mathbf{b}$.

$\dfrac{\partial V_y}{\partial z} = 0$. For the vector potential of the magnetic field **A**, we use pseudovacuum boundary conditions $\dfrac{\partial A_x}{\partial z} = \dfrac{\partial A_y}{\partial z} = A_z = 0$, which corresponds to $B_x = B_y = \dfrac{\partial B_z}{\partial z} = 0$ for the field itself. Set (1) was solved numerically by the control volume approach using two parallel computers of the Interdepartmental Supercomputer Center, Russian Academy of Sciences, for grids 64^3.

Rapid rotation of the hydrodynamic system leads to the appearance of the geostrophic state, when the balance of the pressure gradient and the Coriolis force is fulfilled in the principal expansion order over the Ekman number in the Navier–Stokes equation. The convection has cyclonic form. The horizontal scale of cyclones $l_c = \mathrm{E}^{1/3}L$ [4] can be very small (Fig. 1). It is shown in Fig. 2 that the seed magnetic field is introduced into the system against the background of the already existing developed convection at instant $t = 1$. After the rapid field rise, the system transforms into the quasi-steady state, which is accompanied by a decrease in the convection intensity and close values of the kinetic and magnetic energies:

$$E_{\mathrm{k}} = \frac{\overline{V^2}}{2}, \quad E_{\mathrm{m}} = \frac{1}{2\mathrm{Ro}}\overline{B^2},$$

where the top line means averaging over the volume. Averaging the fields over the horizontal plane and time gives the dependence of characteristics along the vertical coordinate z (Fig. 3). We note that the magnetic field intensity is maximal in the zones of the maximal gradient of the kinetic energy, which corresponds to the increased generation of the magnetic field.

A distinctive feature of the appearing cyclonic convection is the existence of a correlation between velocity field $\mathbf{V}$ and its rotor $\omega = \mathrm{rot}\mathbf{V}$, which is called the hydrodynamic helicity: $\chi^H = \mathbf{V} \cdot \omega$. In the three-dimensional representation, in the absence of external forces (including the Coriolis and Lorenz forces) and dissipation, quantity χ^H averaged over the volume, similarly to kinetic energy E_{k}, is the motion integral of the Navier–Stokes equation. In contrast to the kinetic energy, which is a scalar quantity, χ^H is a pseudoscalar and can change the sign in the volume; therefore, the integral of χ^H over the entire volume can be uninformative. Figure 4 shows the vertical profile of χ^H. The ascending flows that are formed near the lower boundary are twisted in the direction of total rotation Ω, thus providing the positive value of χ^H. The descending cold flows, being spread in the horizontal direction near the boundary, are twisted in the inverse direction, which also corresponds to $\chi^H > 0$. Near the upper boundary, the situation is inverse, and $\chi^H < 0$. This mechanism of formation of hydrodynamic helicity is characteristic of incompressible media; it is observed in planetary dynamo models, where the medium is molten metal [5].

The existence of nonzero mean helicity is closely associated with generation of large-scale magnetic fields [2]. In the absence of mean helicity, the magnetic fields are small-scale [6]. According to the mean-field dynamo theory, electromotive force $\mathscr{E} = \overline{(\mathbf{v} \times \mathbf{b})}$ can be expressed through the mean magnetic field $\overline{\mathbf{B}}$ and α-effect: $\mathscr{E}_i = \alpha_{ij}\overline{B}_j$, where the expansion of the fields into the medium and fluctuation components is used, $\mathbf{V} = \overline{\mathbf{V}} + \mathbf{v}$, $\mathbf{B} = \overline{\mathbf{B}} + \mathbf{b}$. For uniform and isotropic random fields, tensor α_{ij} can be reduced to the scalar function of coordinates $\alpha_0(r)$.

The cornerstone of the theory is the relation between α_0 and χ^H:

$$\alpha_0 = -\frac{\tau}{3}\chi^H, \tag{2}$$

where τ is the characteristic turbulent time. Despite the fact that the assumption on the uniformity and isotropism of the fields is rarely fulfilled for actual objects, relation (2) is often used in the mean-field dynamo theory and gives correct evaluations. Using the results of the three-dimensional simulation of set (1), we can verify the degree of fulfillment of relation (2) in reality by calculating $\alpha_0(z) = \frac{(\mathscr{E} \cdot \overline{\mathbf{B}})}{|\overline{B}^2|}$ and comparing it with the behavior of the mean hydrodynamic helicity (Fig. 4).

If we do not go into detail about the behavior of α_0 near the boundary layers $z = 0, 1$, where dependence (2) is deliberately inapplicable, the spatial distribution of α_0 is in general close to prediction (2): it has the correct sign, the sinusoidal dependence, and changes sign in the center of the volume. The evaluation of the proportionality coefficient gives $\frac{\tau}{3} \sim 10^{-3}$. On the other hand, $\tau \sim \frac{l_d}{E_{\mathrm{k}}^{1/2}} \approx \frac{0.1}{100} = 10^{-3}$, which coincides with the previous evaluation of the mean-field dynamo theory by the order of magnitude.

If we assume that the fields are isotropic and random, but α is the tensor, then $\alpha_{ij} = \alpha_0\delta_{ij}$ and, consequently, $\alpha_0 = \alpha_{ii} = \frac{\mathscr{E}_i}{\overline{B}_i}$. A clearly weaker similarity with theoretical predictions follows from the numerical counting than for the scalar case. One of the causes is an increase in errors at small values of the mean field in the regions of changing its sign. However, the most important fact is that the models of the planar layer with periodic boundary conditions in horizontal directions [7], which are widely used in turbulence, cannot reproduce the mean magnetic field in the vertical direction if we assume that the mean is averaged over the (x, y) plane. The latter follows from solenoidality and periodicity of the magnetic field:

$$\overline{B}_z = \iint\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)dxdy \equiv 0.$$

In this case, either introduction of the intermediate averaging scale $l_d \ll l_m \ll L$ or rejection of the theory is required.

In this work, we consider the simplest relation of mean magnetic field $\overline{\mathbf{B}}$ with pulsations of turbulent velocity field $\mathbf{v}$ and magnetic field $\mathbf{b}$. It is surprising, but despite the violation of isotropism of turbulent fields in geostrophic flows caused by rapid rotation of the system, the obtained α-effect in three-dimensional dynamo models agrees with predictions of the mean-field dynamo theory. A possible explanation of this success is that the vertical component of the mean magnetic field $\overline{B_z} = 0$ in the model, which is omitted from the evaluation of α_0 and coincides with the direction of the revolution axis of the system, compensates for the violation in anisotropy. This approach can be used for testing the magnetic component of the α-effect [8] responsible for stabilization of the rise of the magnetic field for the nonlinear mode.

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